TOPOLOGICAL MIXING FOR SUBSTITUTIONS ON TWO LETTERS

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Abstract. We investigate topological mixing for $\mathbb{Z}$ and $\mathbb{R}$ actions associated with primitive substitutions on two letters. The characterization is complete if the second eigenvalue θ_2 of the substitution matrix satisfies $|\theta_2| \neq 1$. If $|\theta_2| < 1$, then (as is well-known) the substitution system is not topologically weak mixing, so it is not topologically mixing. We prove that if $|\theta_2| > 1$, then topological mixing is equivalent to topological weak mixing, which has an explicit arithmetic characterization. The case $|\theta_2| = 1$ is more delicate, and we only obtain some partial results.

1. Introduction and statement of results

Let X be a compact metric space and let $G = \mathbb{Z}^d$ or $\mathbb{R}^d$ act continuously on X. Let $|g|$ be the distance from g to 0 in some translation-invariant metric.

The dynamical system (X, G) is said to be *topologically mixing* if for any two nonempty open sets $U, V \subset X$, there exists $R > 0$ such that

$$U \cap T_g(V) \neq \emptyset, \quad \text{for all } g \in G,\ |g| \geq R.$$

The dynamical system is *topologically weak mixing* if it has no non-constant continuous eigenfunctions. It is easy to see that topological mixing implies topological weak mixing.

For a symbolic dynamical system, topological mixing is equivalent to the property that for any two allowed blocks W_1 and W_2 there exists $N \in \mathbb{N}$ such that for any $n \geq N$ there is an allowed block $W_1 B W_2$, with $|B| = n$.

Let $\mathcal{T}$ be a tiling of $\mathbb{R}_+$ with a finite set of interval prototiles, and let $X_{\mathcal{T}}$ be the associated tiling space. That is, $X_{\mathcal{T}}$ is the set of all tilings $\mathcal{S}$ of $\mathbb{R}$ such that every patch of $\mathcal{S}$ is the translate of a patch in $\mathcal{T}$. (A *patch* is a tiling of a finite

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interval, and two patches are said to be *equivalent* if each is a translate of the other). The *tiling dynamical system* is the $\mathbb{R}$-action $(X_{\mathcal{T}}, T_g)$ where $T_g(\mathcal{S}) = \mathcal{S} - g$. The topology on $X_{\mathcal{T}}$ has basis given by "cylinder sets" indexed by a patch P and a radius ϵ:

$$X_{P,\epsilon} = \{\mathcal{S} \in X_{\mathcal{T}} : \ \exists y \in (-\epsilon, \epsilon) : \ P - y \subset \mathcal{S}\}.$$

It follows that $(X_{\mathcal{T}}, T_g)$ is topologically mixing if and only if for any two patches $P_1, P_2 \subset \mathcal{T}$, and for any $\epsilon > 0$, there exists $R > 0$ such that for all $g \in \mathbb{R}$, with $|g| \geq R$, there exist $\mathcal{S} \in X_{\mathcal{T}}$ and $y \in (-\epsilon, \epsilon)$, such that

$$P_1 \subset \mathcal{S} \quad \text{and} \quad P_2 - g - y \subset \mathcal{S}.$$

We say that a set $Y \subset \mathbb{R}$ is *eventually dense* in $\mathbb{R}$ if for any $\epsilon > 0$ there exists $R > 0$ such that the ϵ-neighborhood of Y covers $\mathbb{R} \setminus (-R, R)$. An alternative way to state topological mixing for a tiling system is to say that for any allowed patches P_1 and P_2, the set of translation vectors between locations (say, left endpoints) of patches equivalent to P_1 and patches equivalent to P_2 is eventually dense in $\mathbb{R}$.

We will also say that $Y \subset \mathbb{R}_+$ is eventually dense in $\mathbb{R}_+$ if $Y \cup -Y$ is eventually dense in $\mathbb{R}$.

Now we briefly recall the definition of substitutions and associated dynamical systems; see [11, 10] for more details. Let ζ be a substitution on a finite alphabet $\mathcal{A} = \{0, 1, \ldots, m-1\}$, with $m \geq 2$. Recall that ζ is a mapping from $\mathcal{A}$ to $\mathcal{A}^*$, the set of nonempty words in the alphabet $\mathcal{A}$. The substitution ζ is extended to maps (also denoted by ζ) $\mathcal{A}^* \to \mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ by concatenation. The matrix $M = M_\zeta$ associated with the substitution ζ is the $m \times m$ matrix defined by

$$M = (m_{i,j})_{d \times d}, \quad \text{where} \quad m_{i,j} = \ell_i(\zeta(j))$$

where $\ell_i(W)$ is the number of occurrences of letter i in word W. We will always assume that the substitution is *primitive*, that is, there exists k such that all entries of M^k are strictly positive; equivalently, for every $i, j \in \mathcal{A}$, the symbol j occurs in $\zeta^k(i)$. The *substitution space* X_ζ is defined as the set of all two-sided infinite sequences in the alphabet $\mathcal{A}$ whose every word occurs in $\zeta^k(i)$ for k sufficiently large. Then X_ζ is a closed shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$; the $\mathbb{Z}$-action associated with the substitution is (X_ζ, σ) where σ is the left shift. We may assume, without loss of generality, that $\zeta(0)$ starts with 0 (if not, replace ζ with ζ^k for appropriate k and rename the symbols). Then we get a one-sided fixed point of the substitution map $u = \zeta(u) = \lim_{n\to\infty} \zeta^n(0)$, sometimes called the *substitution sequence*. Primitive

substitution $\mathbb{Z}$-actions are uniquely ergodic and minimal, see [11], which implies that every allowed word in the substitution space occurs in u. We denote by $\mathcal{L}(X_\zeta)$ the *language of the substitution*, that is the collection of all allowed words (or equivalently, all subwords of u). In addition, we will always assume that ζ is *aperiodic*, that is, the substitution sequence u is not periodic, which is equivalent to X_ζ being infinite.

Next we recall the definition of the tiling dynamical system associated with the substitution ζ; see [14, 2] for more details. Let $\mathbf{t} = (t_i)_{i\in\mathcal{A}}$ be a strictly positive row vector. We associate to u a tiling $\mathcal{T}$ of $\mathbb{R}_+$ whose prototiles are intervals τ_i of length t_i, for $i = 0, \dots, m-1$, in such a way that 0 is the left endpoint of the tile τ_{u_1}, followed by the copy of τ_{u_2}, etc. *A priori*, some of the tiles may be congruent; then we distinguish them by "labels" from the alphabet $\mathcal{A}$. The tiling space $X_\mathcal{T}$ is defined, as above, as the set of tilings $\mathcal{S}$ of the line $\mathbb{R}$ such that every patch of $\mathcal{S}$ is a translate of a $\mathcal{T}$-patch, and the group $\mathbb{R}$ acts by translation. Observe that this $\mathbb{R}$-action is topologically conjugate to the suspension flow over the $\mathbb{Z}$-action (X_ζ, σ), with the height function equal to t_i on the cylinder corresponding to the symbol i. This system is also minimal and uniquely ergodic.

Mixing properties of general uniquely ergodic systems have been much investigated, and we do not survey this literature here. Many early references can be found in the paper by Petersen and Shapiro [9]. For such systems, measure-theoretic strong mixing implies topological strong mixing, which implies topological weak mixing, and none of the implications can be reversed, see [9]. Mixing properties of primitive substitution $\mathbb{Z}$-actions were studied by Dekking and Keane [3], who proved that they are never strongly mixing, but may be topologically mixing. More recently, topological mixing for substitutions on two symbols was investigated by A. Livshits [5, 6, 7], but a characterization of such systems was still lacking. In another, though related, direction, Host [4] proved that for substitution $\mathbb{Z}$-actions topological weak mixing is equivalent to measure-theoretic weak mixing. For substitution $\mathbb{R}$-actions this is essentially proved in [2, Thm 2.3]. (Alternatively, one can argue as in [14, Thm 4.3], using [2, Lem 2.1] instead of [14, Lem 4.2].) Thus, when writing "weak mixing" we will not specify whether it is in the topological or measure-theoretic category.

We begin with a standard elementary proposition which contains general necessary conditions for topological mixing of substitution systems.

Proposition 1.1. *Let ζ be a primitive aperiodic substitution on the alphabet $\mathcal{A}$.*

(i) *If the $\mathbb{Z}$-action (X_ζ, σ) is topologically mixing, then*

$$GCD\{|\zeta^n(i)| : \ i \in \mathcal{A}\} = 1, \quad \text{for all } n \geq 1. \tag{1.1}$$

(ii) *Let $(t_i)_{i\in\mathcal{A}}$ be a vector of tile lengths. If the $\mathbb{R}$-action $(X_\mathcal{T}, T_g)$ is topologically mixing, then*

$$\text{There exist } i, j \in \mathcal{A} \text{ such that } t_i/t_j \text{ is irrational.} \tag{1.2}$$

Proof. (i) Suppose that for some $m \in \mathbb{N}$, the numbers $|\zeta^m(i)|$, $i \in \mathcal{A}$, have a common factor p. Primitive aperiodic substitutions are known to be bilaterally recognizable [8]. This implies that if an allowed word W is sufficiently long, then it is uniquely determined where the subwords $\zeta^m(i)$ occur within W (after some "buffer" of uniformly bounded length is removed from the beginning and the end) in any occurrence of W in the substitution sequence u. Then it follows that the distance between any two occurrences of W is divisible by p, contradicting topological mixing.

(ii) Clearly, the left endpoints of all tiles of the tiling $\mathcal{T}$ belong to the $\mathbb{Z}$-module generated by the t_i where $i \in \mathcal{A}$. If the lengths are all rationally related, then this $\mathbb{Z}$-module is a discrete subset of $\mathbb{R}$, which contradicts topological mixing for the $\mathbb{R}$-action. □

From now on we consider primitive aperiodic substitutions on 2 symbols only. We will use the basic facts of Perron-Frobenius theory, see, e.g. [13]. Let θ_1 be the Perron-Frobenius eigenvalue of the substitution matrix M and let θ_2 be the second eigenvalue. Our main result is

Theorem 1.2. *Suppose that $|\theta_2| > 1$. Then*

(i) *the $\mathbb{Z}$-action (X_ζ, σ) is topologically mixing if and only if (1.1) holds;*

(ii) *the $\mathbb{R}$-action $(X_\mathcal{T}, T_g)$ is topologically mixing if and only if (1.2) holds (that is, if $t_1/t_0 \notin \mathbb{Q}$).*

Remarks.

1. If $|\theta_2| < 1$, then both the $\mathbb{Z}$-action and the $\mathbb{R}$-action associated with the substitution have nontrivial continuous eigenfunctions [4, 2], so they are not topologically

mixing. The case $|\theta_2| = 1$ is more subtle. Dekking and Keane [3] considered the following two substitutions:

$$\zeta_1(0) = 001, \quad \zeta_1(1) = 11100; \qquad \zeta_2(0) = 001, \quad \zeta_2(1) = 11001.$$

They have the same substitution matrix with eigenvalues 4 and 1. In [3] it is proved that the $\mathbb{Z}$-action associated with ζ_1 is topologically mixing, whereas the one associated with ζ_2 is not (the latter actually goes back to [9]). We show (Theorem 1.4(ii) and section 5) that the same is true for the corresponding $\mathbb{R}$-actions (with irrational ratio of tile lengths).

2. Partial results in the direction of conclusion (i) were obtained by A. Livshits [5, 6, 7], and in fact some of our methods are similar to his. He conjectured that (i) holds.

3. As far as we are aware, topological mixing for substitution $\mathbb{R}$-actions has not been considered before. A special choice of the tile lengths is the one arising from the Perron-Frobenius eigenvector of the substitution matrix. Then we get a (geometrically) self-similar tiling of the half-line.

4. Given a substitution, it is straighforward to check condition (1.1). Let $r = (1, \ldots, 1)$. Then $rM^n = (|\zeta^n(0)|, \ldots, |\zeta^n(m-1)|)$. The question is whether, for any prime p, this vector is eventually zero mod p. If p does not divide the determinant of M, then M is invertible mod p, and rM^n can never equal zero mod p. Thus we need only consider primes that divide the determinant of M. For each of these, the sequence of vectors $\{rM^n \pmod p\}$ takes on at most p^m values, hence starts repeating after at most p^m terms, so we need only examine those first p^m terms.

5. Results similar to Theorem 1.2 are known for weak mixing [4, 2]. The difference is that $|\theta_2| \geq 1$ (as opposed to $|\theta_2| > 1$), together with (1.1) or (1.2), implies weak mixing. Thus

Corollary 1.3. *Consider a tiling dynamical system arising from a primitive substitution on two symbols. If $|\theta_2| \neq 1$, then the $\mathbb{R}$ action is topologically mixing if and only if it is weak mixing.*

Although a complete description of substitutions with $|\theta_2| = 1$ remains open, there is something that we can say. The following theorem constitutes one of the two main steps in the proof of Theorem 1.2, and it deals with an arbitrary primitive substitution on two symbols. Consider all words W in $\mathcal{L}(X_\zeta)$ of length $|W| = n$.

Let

$$a(n) = \min \ell_0(W), \qquad\qquad b(n) = \max \ell_0(W)$$

be the minimum and maximum number of 0's in allowed words of length n. We will need the condition

$$b(n) - a(n) \to \infty, \quad \text{as } n \to \infty, \tag{1.3}$$

which is closely related to the "growth of excess" condition that appeared in [3] and [5, 6, 7].

Theorem 1.4. **(i)** *If (1.1) is satisfied, then the $\mathbb{Z}$-action corresponding to the substitution is topologically mixing if and only if (1.3) holds.*

(ii) *If (1.2) is satisfied, then the $\mathbb{R}$-action corresponding to the substitution is topologically mixing if and only if (1.3) holds.*

The next proposition is the second ingredient of the proof of Theorem 1.2.

Proposition 1.5. *Suppose that the substitution satisfies the condition $|\theta_2| > 1$. Then there exists a constant $c_1 > 0$ such that*

$$b(n) - a(n) \ge c_1 n^\alpha, \quad \textit{where } \alpha = \log|\theta_2|/\log\theta_1 \in (0,1).$$

It is clear that Theorem 1.2 will follow once we prove Theorem 1.4 and Proposition 1.5.

The difference between topological mixing and weak mixing is related to the difference between lim-sup and lim-inf. Consider the quantities $\sup_{n<N}(b(n)-a(n))$ and $\inf_{n>N}(b(n)-a(n))$. If $|\theta_2| > 1$, then both of these quantities grow as N^α and the system is both topologically mixing and weak mixing. If $|\theta_2| < 1$, then both of these quantities are bounded and the system is neither topologically mixing nor weak mixing. If $|\theta_2| = 1$, then, as can be shown, the sup grows as $\log(N)$ (implying weak mixing) but the inf may or may not grow.

2. Preliminaries

For a word W consisting of 0's and 1's, recall that $\ell_i(W)$ denotes the number of letters i in W. The column vector $\ell(W) = (\ell_0(W), \ell_1(W))^T$ (where ^{T} denotes transpose) is called the *population vector* of the word W. By the definition of the substitution matrix, for any word V in the alphabet $\mathcal{A}$,

$$\ell(\zeta(V)) = M\,\ell(V). \tag{2.1}$$

Note that the length of a word V is $|V| = (1,1)\cdot \ell(V)$. In a tiling, the length of a patch corresponding to the word V (also called the *tiling length* of V and denoted $|V|_{\mathcal{T}}$) is $(t_0, t_1)\cdot \ell(V)$. Recall that $\mathcal{L}(X_\zeta)$ denotes the *language* of the substitution subshift, that is, the set of all words that occur in the subshift. Consider

$$\Phi(X_\zeta) := \{\ell(W) : \ W \in \mathcal{L}(X_\zeta)\}.$$

Further, consider the set $\Gamma(X_\zeta)$ of all the points $\ell(u[1,j])$ for $j \geq 0$ (for $j = 0$ we just get the origin). Observe that

$$\Phi(X_\zeta) = (\Gamma(X_\zeta) - \Gamma(X_\zeta)) \cap \mathbb{Z}_+^2.$$

When the substitution is fixed, we drop X_ζ from the notation and write just $\mathcal{L}, \Phi, \Gamma$. Connecting the consecutive points of Γ we obtain a polygonal curve $\widetilde{\Gamma}$, which gives a nice visual representation of the substitution sequence u. This curve starts at the origin and goes into the 1st quadrant, with edges going up or to the right along the standard grid. Clearly $(1,0)^T \in \Gamma$ since u starts with 0.

Lemma 2.1. **(i)** $\Phi = \bigcup_{n\geq 1}\{(i, n-i)^T : \ a(n) \leq i \leq b(n)\}$.

(ii) $0 \leq b(n+1) - b(n) \leq 1;\ \ 0 \leq a(n+1) - a(n) \leq 1$.

(iii) *If* $(i,j)^T, (i',j')^T \in \Phi$ *and* $i \leq i', j \geq j'$, *then* $(k,m)^T \in \Phi$ *whenever* $i \leq k \leq i'$ *and* $j' \leq m \leq j$.

Proof. (i) Consider the population vectors of all words of length n. They include $(a(n), n-a(n))^T$ and $(b(n), n-b(n))^T$ and hence all intermediate ones since $|\ell_0(u[i, i+n-1]) - \ell_0(u[i+1, i+n])| \leq 1$.

(ii) is obvious from the definition.

(iii) follows easily from (i) and (ii). The details are left as an exercise. $\square$

By Lemma 2.1(i), the "upper envelope" of Φ is the set $\{(a(n), n-a(n)) : \ n \geq 1\} \cup (0,0)$, and the "lower envelope" of Φ is the set $\{(b(n), n-b(n)) : \ n \geq 1\} \cup (0,0)$. We connect the consecutive points of these "envelopes" to obtain two polygonal curves and consider the set $\widetilde{\Phi}$ between them. It is a kind of "strip" with polygonal edges; by definition, $\Phi = \widetilde{\Phi} \cap \mathbb{Z}_+^2$.

Example 2.2. Let $\zeta(0) = 011,\ \zeta(1) = 0$. The matrix of the substitution is $M = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, with the eigenvalues $\theta_1 = 2,\ \theta_2 = -1$. The fixed point of the

substitution is

$$u = 0110001101101100011000110001101101100011011...$$

This fixed point is represented by $\widetilde{\Gamma}$, shown as the thick line in Figure 1. The region $\widetilde{\Phi}$ is shaded. We will show in Section 5 that for this substitution $\sup(b(n)-a(n)) = \infty$ but $\liminf(b(n) - a(n)) \leq 2$.

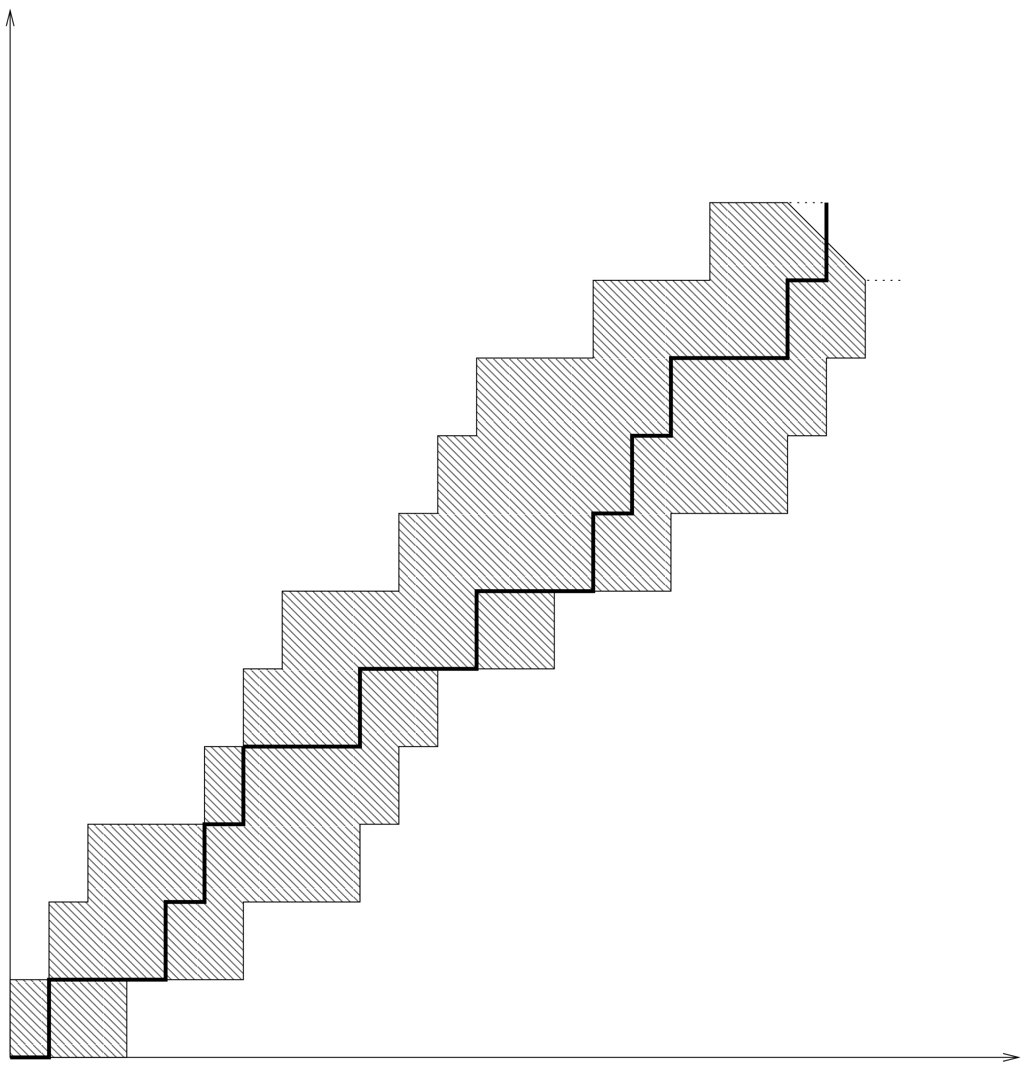

FIGURE 1. The sets $\widetilde{\Gamma}$ and $\widetilde{\Phi}$

Definition 2.3. For any $\gamma > 0$, we define the width of $\widetilde{\Phi}$ in the direction of $(-\gamma, 1)$ at the level $r > 0$ to be the length of the intersection of $\widetilde{\Phi}$ with the line $x + \gamma y = r$.

For a tiling system with $\mathbf{t} = (1, \gamma)$, the width of $\widetilde{\Phi}$ in the direction of $(-\gamma, 1)$ at the level r governs the number of distinct population vectors whose tiling lengths are approximately r. In particular, for γ irrational, the set of tiling lengths of allowed words is asymptotically dense if and only if the width of $\widetilde{\Phi}$ goes to infinity as $r \to \infty$. This geometric observation is at the heart of the proofs of Propositions 3.3 and 3.5, below.

Observe that $\sqrt{2}(b(n) - a(n))$ is the width of $\widetilde{\Phi}$ in the direction of $(-1, 1)$ at the level n (since the points $(a(n), n - a(n))^T$ and $(b(n), n - b(n))^T$ lie on the line $x + y = n$). The following lemma shows that the asymptotic behavior of the width for large levels does not depend on the direction, and hence is determined by the large-n behavior of $b(n) - a(n)$.

Lemma 2.4. *For any $\gamma_1, \gamma_2 > 0$ there exist constants $K_{\gamma_1,\gamma_2}, C_{\gamma_1,\gamma_2} > 1$ such that if $\widetilde{\Phi}$ has width L_1 in the direction of $(-\gamma_1, 1)^T$ at the level r_1, then there is*

$$r_2 \in (K^{-1}_{\gamma_1,\gamma_2} r_1, K_{\gamma_1,\gamma_2} r_1)$$

such that the width L_2 of $\widetilde{\Phi}$ in the direction of $(-\gamma_2, 1)^T$ at the level r_2 satisfies

$$C^{-1}_{\gamma_1,\gamma_2} L_1 \le L_2 \le C_{\gamma_1,\gamma_2} L_1.$$

Proof. This is a simple geometric fact. From Lemma 2.1(iii), $\tilde{\Phi}$ contains a rectangle R_1 with sides parallel to the axes, with aspect ratio γ_1 and whose diagonal is on the line $x + \gamma_1 y = r_1$ and has length L_1. For a well-chosen r_2, there is a rectangle R_2 of aspect ratio γ_2 contained in this rectangle, whose diagonal is on the line $x + \gamma_2 y = r_2$, and whose diagonal length is proportional to L_1. Clearly $r_2 \in (K^{-1} r_1, K r_1)$ for some constant K depending only on γ_1 and γ_2. $\square$

Let e_i be eigenvectors of M corresponding to θ_i, for $i = 1, 2$. By the Perron-Frobenius theory, e_1 has components of the same sign and e_2 has components of opposite signs. We choose e_1 to be strictly positive and e_2 such that $(e_2)_x < 0$. Denote

$$\gamma := (e_1)_y/(e_1)_x, \qquad \alpha = \log|\theta_2|/\log\theta_1.$$

For $w = (w_x, w_y) \in \mathbb{R}^2$, we define the maps

$$\pi_\gamma(w) = w_x + \gamma w_y, \qquad \pi_\gamma^\perp(w) = w_y - \gamma w_x. \tag{2.2}$$

Up to an overall factor of $\sqrt{1+\gamma^2}$, these give the (signed) length of the projections of w onto lines parallel and perpendicular to e_1, respectively. Note that $\pi_\gamma^\perp(e_1) = 0$ and $\pi_\gamma^\perp(e_2) > 0$.

3. Proof of Theorem 1.4

In this section we prove a series of propositions. The first three show that (1.3) is a necessary condition for topological mixing. The last three show that (1.3), together with (1.1) or (1.2), implies mixing. The next proposition follows from [1, Theorem 22], which contains precise bounds on $\limsup(b(n)-a(n))$. We provide a proof for completeness.

Proposition 3.1. *Suppose that a primitive aperiodic substitution ζ on two letters has the second eigenvalue θ_2 satisfying $|\theta_2| \geq 1$. Then $\limsup(b(n)-a(n)) = \infty$.*

Proof. For any n, $b(n) \geq n\gamma/(\gamma+1) \geq a(n)$, since the average density of 0's is $\gamma/(\gamma+1)$. To show that $\limsup(b(n)-a(n)) = \infty$, it suffices to show that either $b(n) - \frac{n\gamma}{\gamma+1}$ or $\frac{n\gamma}{\gamma+1} - a(n)$ is unbounded. This is equivalent to showing that there are vectors $w \in \Gamma - \Gamma$ with $|\pi_\gamma^\perp(w)|$ arbitrarily large.

For $|\theta_2| > 1$ this is easy, since $\pi_\gamma^\perp(Mw) = \theta_2 \pi_\gamma^\perp(w)$ and since $\pi_\gamma^\perp(1,0)^T \neq 0$. All that remains are the cases $\theta_2 = \pm 1$. By squaring the substitution, we can assume that $\theta_2 = 1$.

We can choose $i \in \mathcal{A}$ so that ii is allowed, since otherwise the substitution is periodic. Then, by primitivity, there exists $p \in \mathbb{N}$ such that $\zeta^p(i) = V_1 ii V_2$ for some words V_1, V_2. Note that at least one of $\ell(V_2), \ell(iV_2)$ is not a multiple of e_1. Thus we can write

$$\zeta^p(i) = W_1 i W_2$$

where $\ell(W_2) = a_1 e_1 + a_2 e_2$, with $a_2 \neq 0$. Then

$$\zeta^{2p}(i) = \zeta^p(W_1) W_1 i W_2 \zeta^p(W_2),$$

and iterating this we obtain that

$$\forall\, m \geq 1,\ U_m := W_2 \zeta^p(W_2) \ldots \zeta^{mp}(W_2) \in \mathcal{L}(X_\zeta).$$

It follows that

$$\pi_\gamma^\perp(\ell(U_m)) = a_2 \pi_\gamma^\perp(e_2) m, \tag{3.1}$$

which can be made arbitrarily large. $\square$

Proposition 3.2. *Let S be any subshift on 2 letters (not necessarily generated by a substitution). Let $b(n)$ be the maximum number of 0's in words of length n, and let $a(n)$ be the minimum. If $\sup(b(n)-a(n)) > \liminf(b(n)-a(n))$, then the $\mathbb{Z}$ action on S is not topologically mixing.*

Proof. Let V_n (resp. W_n) be a word of length n that maximizes (resp. minimizes) the number of 0's. Suppose that $m > n$ and that $b(m) - a(m)$ is strictly less than $b(n) - a(n)$. Then words of the form $V_n U W_n$, where U is a spacer of size $m - n$, can never occur, as $V_n U$ and $U W_m$ differ in population by more than $b(m) - a(m)$. That is, the cylinder set based on V_n, and the translate by m of the cylinder set based on W_n, do not intersect.

Now pick n such that $b(n) - a(n)$ is larger than the lim inf. Since there are arbitrarily large values of m for which $b(n) - a(n) > b(m) - a(m)$, the cylinders on V_n and W_n do not mix. □

Proposition 3.3. *Let $X_{\mathcal{T}}$ be a tiling space based on a substitution on two letters. If* $\liminf\,(b(n) - a(n)) < \infty$*, then the $\mathbb{R}$ action on $X_{\mathcal{T}}$ is not topologically mixing.*

Proof. We can assume without loss of generality that the tile lengths are $t_0 = 1$, $t_1 = \beta > 0$. Then $\pi_\beta(\Phi)$ is the set of tiling lengths of words in $\mathcal{L}(X_\zeta)$ (here π_β is defined by $\pi_\beta(w_x, w_y) = w_x + \beta w_y$ as in (2.2)). It is enough to show that $\pi_\beta(\Phi)$ is not eventually dense. Since $\liminf(b(n) - a(n)) < \infty$, the width of the set $\widetilde{\Phi}$ in the direction $(-1, 1)$ is bounded at a sequence of levels $k_n \to \infty$. Then the width of $\widetilde{\Phi}$ in the direction $(-\beta, 1)$ is bounded at a sequence of levels $k_n' \to \infty$, by Lemma 2.4. Since Φ is the set of lattice points in $\widetilde{\Phi}$, it follows that $\pi_\beta(\Phi)$ is not eventually dense. □

Propositions (3.1, 3.2, 3.3) imply half of Theorem 1.4, namely that (1.3) is a necessary condition for topological mixing. The remainder of this section is to prove sufficiency.

Denote

$$\Phi' = \{\ell(0W) :\ 0W1 \in \mathcal{L}\}. \tag{3.2}$$

Lemma 3.4.

$$\Phi \setminus \{(a(n), n - a(n)) :\ n \geq 1\} \subset \Phi'.$$

Proof. Fix n and consider $f_n(j) = \ell_0(u[j, j+n-1])$. We have $|f_n(j+1) - f_n(j)| \leq 1$, with $f_n(j+1) - f_n(j) = -1$ if and only if $u_j = 0$ and $u_{j+n} = 1$. Since u is uniformly recurrent (the substitution being primitive), $a(n) = \min_j f_n(j)$ and $b(n) = \max_j f_n(j)$ are achieved infinitely often. It follows that

$$\forall k \in [a(n)+1, b(n)] \cap \mathbb{Z},\ \exists j \geq 1 :\quad u_j = 0, u_{j+n} = 1,\quad \ell_0(u[j, j+n-1]) = k. \tag{3.3}$$

This implies the desired statement, in view of Lemma 2.1. □

Proposition 3.5. *Let $X_{\mathcal{T}}$ be a tiling space based on a substitution on two letters. If* $\liminf(b(n) - a(n)) = \infty$ *and the ratio of tile lengths is irrational, then the* $\mathbb{R}$ *action on* T *is topologically mixing.*

Proof. By the definition of our tiling space, all patches are determined by allowed words of the substitution space. Recall that the tiling length of a word V is

$$|V|_{\mathcal{T}} = \ell_0(V)t_0 + \ell_1(V)t_1 = \mathbf{t} \cdot \ell(V),$$

where $\mathbf{t} = (t_0, t_1)$ is the vector of prototile lengths. To prove topological mixing, we need to show that for any allowed words W_1, W_2, the set

$$\Psi(W_1, W_2) := \{|W_1 V|_{\mathcal{T}} :\ W_1 V W_2 \in \mathcal{L}\}$$

is eventually dense in $\mathbb{R}_+$. Since ζ is a primitive substitution, there exists m such that $\zeta^m(0)$ contains W_1 and $\zeta^m(1)$ contains W_2. Thus, it is enough to prove the claim for $W_1 = \zeta^m(0)$ and $W_2 = \zeta^m(1)$ for an arbitrary $m \in \mathbb{N}$. Clearly,

$$\Psi(\zeta^m(0), \zeta^m(1)) \supset \Xi_m := \{|\zeta^m(0W)|_{\mathcal{T}} :\ 0W1 \in \mathcal{L}\}.$$

We have

$$|\zeta^m(0W)|_{\mathcal{T}} = \mathbf{t}\ell(\zeta^m(0W)) = \mathbf{t}M^m\ell(0W).$$

Thus,

$$\Xi_m = \{\mathbf{t}M^m z :\ z \in \Phi'\}$$

by (3.2). Since t_1/t_0 is irrational and M is an invertible integer matrix, it is easy to see that $\mathbf{t}M^m$ is a vector with irrational ratio of components. (If M were not invertible, then θ_2 would be zero and $b(n) - a(n)$ would be bounded.) Let γ_m denote this ratio. Up to an overall scale, Ξ_m is a projection of Φ' onto a line with slope γ_m. By Lemma 3.4, $\Phi' \supset \mathbb{Z}_+^2 \cap int(\widetilde{\Phi})$. By assumption and Lemma 2.4, the width of $\widetilde{\Phi}$ in the direction $(-\gamma_m, 1)$ tends to infinity, which implies the desired statement. □

Proposition 3.6. *Let ζ be a primitive substitution on two letters. If conditions (1.1) and (1.3) are met, then the* $\mathbb{Z}$ *action on* X_ζ *is topologically mixing.*

Proof. Let W_1, W_2 be any allowed words. We need to show that there exists $N \in \mathbb{N}$ such that for all $k \geq N$ there is an allowed word $W_1 V W_2$ with $|W_1 V W_2| = k$. Since ζ is a primitive substitution, there exists m such that $\zeta^m(0)$ contains W_1 and

$\zeta^m(1)$ contains W_2. Thus, it is enough to prove the claim for $W_1 = \zeta^m(0)$ and $W_2 = \zeta^m(1)$.

By assumption (1.1), we can find integers r, s such that

$$r|\zeta^m(0)| + s|\zeta^m(1)| = 1. \tag{3.4}$$

Thus, if two words V and V' have $\ell(V) - \ell(V') = (r, s)^T$, then $|\zeta^m(V)| - |\zeta^m(V')| = 1$. We will demonstrate the existence of a sequence of words V_n, each beginning with 0 and ending with 1, such that every integer is within a bounded error of a length $|\zeta^m(V_n)|$. Once the width of $\tilde{\Phi}$ in the $(-s/r, 1)$ direction is large enough, there will exist other words $V_{n,j}$, also beginning with 0 and ending with 1, whose population vectors differ from that of V_n by $(jr, js)^T$, so that $|\zeta^m(V_{n,j})| = |\zeta^m(V_n)| + j$. For every sufficiently large integer k, we can then pick n and j such that $|\zeta^m(V_{n,j})| = k$.

For each n sufficently large that $b(n) - a(n) \geq 2$, let

$$p_n := \lfloor (a(n) + b(n))/2 \rfloor, \quad q_n := n + 1 - p_n,$$

and let V_n be a word, beginning with 0 and ending with 1, with population vector (p_n, q_n). Let $u_n = |\zeta^m(V_n)|$. It easily follows from Lemma 2.1(ii) that $|p_{n+1} - p_n| \leq 1$ and $|q_{n+1} - q_n| \leq 1$, hence

$$|u_{n+1} - u_n| \leq |\zeta^m(0)| + |\zeta^m(1)|.$$

Note also that $u_n \to \infty$. Thus, for any k sufficiently large we can find n such that

$$u_n \leq k \leq u_{n+1} \leq u_n + |\zeta^m(0)| + |\zeta^m(1)|. \tag{3.5}$$

If k is sufficiently large, then n is large as well, and we can make sure that

$$b(n) - a(n) \geq 5 + A, \quad \text{where} \;\; A := \max\{|r|, |s|\}(|\zeta^m(0)| + |\zeta^m(1)|), \tag{3.6}$$

and hence that $b(n+1) - a(n+1) \geq 4 + A$. Now consider the points $(p_n + r(k - u_n), q_n + s(k - u_n))$ and $(p_{n+1} + r(k - u_{n+1}), q_{n+1} + s(k - u_{n+1}))$. Since $k - u_n$ and $k - u_{n+1}$ have opposite signs, one of these lies above (or on) the line traced by the points (p_n, q_n) and the other lies below (or on), and their distance is less than the width of $\tilde{\Phi}$. Hence at least one of these points lies in the interior of $\tilde{\Phi}$ and there either exists a word $V_{n,k-u_n}$ or a word $V_{n+1,k-u_{n+1}}$, beginning with 0 and ending with 1. Either way, ζ^m applied to this word has length exactly k.

□

This completes the proof of Theorem 1.4.

4. Proof of Proposition 1.5

Throughout this section we assume that $|\theta_2| > 1$. Since the system only depends on u, we can square the substitution and assume that $\theta_2 > 1$. We begin with simple bounds on the numbers of 0s and 1s in substituted letters. From these we show that $b(n) - a(n)$ is bounded above by a constant times n^α, where $\alpha = \log|\theta_2|/\log\theta_1$. These upper bounds, combined with an "intermediate value theorem" argument, then yield lower bounds on $b(n) - a(n)$.

Lemma 4.1. *There exist positive constants L_1, L_1', L_2, L_2' such that for all $k \geq 1$,*

$$L_1\theta_1^k \leq \ell_0(\zeta^k(i)) \leq L_1'\theta_1^k, \quad i = 0, 1, \tag{4.1}$$

and

$$L_2|\theta_2|^k \leq |\pi_\gamma^\perp(\ell(\zeta^k(i)))| \leq L_2'|\theta_2|^k = L_2'\theta_1^{k\alpha}, \quad i = 0, 1. \tag{4.2}$$

Moreover, if $\theta_2 > 0$, then $\pi_\gamma^\perp(\ell(\zeta^k(0))) < 0$ and $\pi_\gamma^\perp(\ell(\zeta^k(1))) > 0$.

Proof. Fix $i \in \{0, 1\}$ and write $\ell(i) = a_1^{(i)}e_1 + a_2^{(i)}e_2$. By (2.1),

$$\ell(\zeta^k(i)) = M^k(\ell(i)) = a_1^{(i)}\theta_1^k e_1 + a_2^{(i)}\theta_2^k e_2.$$

The estimates (4.1) hold since $|\theta_2| < \theta_1$ (note that $a_1^{(i)} > 0$ since $\ell(i)$ is a non-negative vector). The estimates (4.2) follow from the fact that $\pi_\gamma^\perp(e_1) = 0$. To verify the last statement, we observe that $\ell(0) = (1, 0)^T$, so $a_2^{(i)} < 0$ by our choice of eigenvectors. On the other hand, $\ell(1) = (0, 1)^T$, so $a_2^{(i)} > 0$. $\square$

Thanks to Lemma 2.4, the following estimate is tantamount to an upper bound on $b(n) - a(n)$.

Lemma 4.2. *Suppose that $|\theta_2| > 1$. Then there is a constant $C_2 > 0$, depending on ζ, such that for any $w \in \Phi$ we have*

$$|\pi_\gamma^\perp(w)| \leq C_2 w_x^\alpha.$$

Proof. A vector w is in Φ if and only if there is a word $V \in \mathcal{L}$ such that $\ell(V) = w$. Since V occurs in $u = \zeta(u)$, we can write V in the following form (sometimes called the "accordion form"):

$$V = s_1\zeta(s_2)\ldots\zeta^{k-1}(s_{k-1})\zeta^k(s_k)\zeta^k(p_k)\zeta^{k-1}(p_{k-1})\ldots\zeta(p_2)p_1$$

where s_j and p_j are respectively suffixes and prefixes (possibly empty) of the words $\zeta(i)$, $i \in \mathcal{A}$. Note that the number of possible words s_j and p_j is finite (at most $|\zeta(0)| + |\zeta(1)|$). We have for any word W,

$$\ell(\zeta^n(W)) = M^n(\ell(W)) = b_1\theta_1^n e_1 + b_2\theta_2^n e_2 \quad \text{for all} \quad n \geq 1,$$

for some constants b_1, b_2. Since $\pi_\gamma^\perp(e_1) = 0$, we obtain

$$|\pi_\gamma^\perp(w)| \leq \text{const} \cdot \sum_{j=0}^{k} |\theta_2|^j < \text{const} \cdot |\theta_2|^k/(|\theta_2| - 1),$$

with the constant depending on the substitution, but not on w. On the other hand, $w_x = \ell_0(V) \geq \text{const}' \cdot \theta_1^k$ by (4.1), and the statement of the lemma follows, since $\theta_1^\alpha = |\theta_2|$. □

Proposition 4.3. *There is a constant $C_3 > 0$ such that for any $w \in \mathbb{Z}_+^2$ satisfying*

$$-C_3 w_x^\alpha < \pi_\gamma^\perp(w) < 0, \tag{4.3}$$

we have $w \in \Phi$.

Proof of Proposition 1.5 assuming Proposition 4.3. The latter implies that the width of $\widetilde{\Phi}$ in the direction of $(-\gamma, 1)^T$ at the level r is bounded below by $\text{const} \cdot r^\alpha$ for any $r > 0$. Then Lemma 2.4, together with the observation preceding it, implies the desired estimate. □

For the proof of Proposition 4.3 we need a simple geometric lemma, a kind of "Intermediate value theorem." We say that a point $z \in \mathbb{Z}_+^2$ is above (resp. below) Γ if $z \notin \Gamma$ and there are no points of Γ directly above (resp. below) it. There is a natural linear order on Γ, with $(0,0)$ being the minimal element, and the $(n+1)$st point always one unit above or to the right from the nth point. Thus, every integer lattice point in the first quadrant either belongs to Γ, or is above Γ, or is below Γ.

Lemma 4.4. *If $w \in \mathbb{Z}_+^2$ and there exist $z, z' \in \Gamma$ such that $z + w$ is below Γ and $z' + w$ is above Γ, then $w \in \Gamma - \Gamma$.*

The proof is straightforward, since when we move from a point z on Γ to the next one, $z + w$ cannot "jump" from being below Γ to being above Γ, or vice versa.

Proof of Proposition 4.3. Let w be a vector satisfying (4.3). First we claim that there exists $z \in \Gamma$ such that $z + w$ is on or below Γ. Suppose this is not true. Since

$z_0 := (0,0) \in \Gamma$, we have that $w = z_0 + w$ is above Γ, hence there exists $z_1 \in \Gamma$ with

$$(z_1)_x = w_x, \qquad (z_1)_y < w_y.$$

Since $z_1 + w$ is above Γ, there exists $z_2 \in \Gamma$ with $(z_2)_x = 2w_x$ and $(z_2)_y < 2w_y$. Iterating this procedure, we obtain a sequence $z_m \in \Gamma$ such that

$$(z_m)_x = mw_x, \qquad (z_m)_y < mw_y.$$

This implies that

$$\pi_\gamma^\perp(z_m) < m\pi_\gamma^\perp(w) = -\frac{|\pi_\gamma^\perp(w)|}{w_x}(z_m)_x,$$

while Lemma 4.2 implies that $\pi_\gamma^\perp(z_m) \geq -C_2(z_m)_x^\alpha$. This is a contradiction for m sufficiently large. Notice that here we only used that $\pi_\gamma^\perp(w) < 0$.

In view of Lemma 4.4, it remains to prove that there exists $z' \in \Gamma$ such that $z' + w$ is on or above Γ. Suppose there is no such z'. For $k \geq 1$ let $\xi_k := \ell(\zeta^k(0))$, which is a point on Γ by definition. Recall that in Lemma 4.1 we showed that $L_1\theta_1^k \leq (\xi_k)_x \leq L_1'\theta_1^k$ and

$$\pi_\gamma^\perp(\xi_k) \leq -L_2\theta_2^k = -L_2\theta_1^{k\alpha}. \tag{4.4}$$

Let

$$L_3 := \left(\frac{L_2}{2C_2\theta_1^\alpha}\right)^{1/\alpha}, \tag{4.5}$$

where C_2 is from Lemma 4.2, and

$$C_3 := C_2L_3/L_1'. \tag{4.6}$$

Further, let k be the integer satisfying

$$L_3\theta_1^k \leq w_x < L_3\theta_1^{k+1}. \tag{4.7}$$

We can assume that w_x is sufficiently large, so that $k \geq 1$, since for small w_x the statement of the proposition is obviously true, perhaps with a different constant. Since $z_0 := (0,0) \in \Gamma$, the point $w = z_0 + w$ is below Γ by our assumption, so there exists $z_1 \in \Gamma$ such that

$$(z_1)_x = w_x, \qquad (z_1)_y > w_y.$$

Iterating this, we obtain $z_m \in \Gamma$ such that

$$(z_m)_x = mw_x, \qquad (z_m)_y > mw_y. \tag{4.8}$$

Let m be the largest integer satisfying $mw_x \le (\xi_k)_x$, so that

$$(\xi_k - mw)_x < w_x. \tag{4.9}$$

We have

$$\pi_\gamma^\perp(\xi_k) = \pi_\gamma^\perp(\xi_k - z_m) + \pi_\gamma^\perp(z_m). \tag{4.10}$$

Now,

$$|\pi_\gamma^\perp(\xi_k - z_m)| \le C_2|(\xi_k - z_m)_x|^\alpha = C_2(\xi_k - mw)_x^\alpha < C_2 w_x^\alpha < C_2 L_3^\alpha \theta_1^{(k+1)\alpha}, \tag{4.11}$$

by Lemma 4.2, (4.9) and (4.7). On the other hand,

$$\begin{aligned}
\pi_\gamma^\perp(z_m) &> m\pi_\gamma^\perp(w) \\
&> -C_3 m w_x^\alpha \\
&> -C_3 m (L_3\theta_1^{k+1})^\alpha \\
&\ge -C_3 \frac{(\xi_k)_x}{w_x} L_3^\alpha \theta_1^{(k+1)\alpha} \\
&\ge -C_3 \frac{L_1'\theta_1^k}{L_3\theta_1^k} L_3^\alpha \theta_1^{(k+1)\alpha} \\
&= -C_3 L_1' L_3^{\alpha-1}\theta_1^{(k+1)\alpha} = -C_2 L_3^\alpha \theta_1^{(k+1)\alpha}.
\end{aligned}$$

Above, we used (4.8) in the 1st line, (4.3) in the 2nd line, (4.7) in the 3d line, the definition of m in the 4th line, (4.1) and (4.7) in the 5th line, and (4.6) in the last line. Combined with (4.10) and (4.11), the last inequality yields

$$\pi_\gamma^\perp(\xi_k) > -2C_2 L_3^\alpha \theta_1^{(k+1)\alpha},$$

contradicting (4.4) in view of (4.5). □

5. Proofs of other results, concluding remarks, and open questions

Example 5.1. Consider the "period doubling" substitution on two letters $\{0, 2\}$: $\zeta(0) = 02$, $\zeta(2) = 00$, whose matrix $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 2 and -1.

Since each $\zeta(i)$ begins with 0 and has length 2, every word of length $2n$ is either ζ applied to a word of length n, or is obtained from such a word by deleting the initial 0 and adding a 0 at the end. In particular, the population vector of each word of length $2n$ is M times the population vector of a word of length n, from which we infer that $b(2n) - a(2n) = b(n) - a(n)$. In particular, $b(2^n) - a(2^n) = b(1) - a(1) = 1$.

A word of length 2^n has either $\lfloor 2^{n+1}/3 \rfloor$ 0's and $\lfloor 2^n/3 \rfloor + 1$ 2's or $\lfloor 2^{n+1}/3 \rfloor + 1$ 0's and $\lfloor 2^n/3 \rfloor$ 2's.

This subshift is neither weak mixing nor topologically mixing, as condition (1.1) is not met. Since $\liminf(b(n) - a(n)) = 1$, the $\mathbb{R}$ action on a tiling space based on this substitution will not be topologically mixing. However, the $\mathbb{R}$ action will be weak mixing if the ratio of tile lengths is irrational [2].

Example 5.2. (= Example 2.2). Consider the "modified period doubling substitution" (MPD) $\zeta(0) = 011$, $\zeta(1) = 0$. The matrix of the substitution is $M = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, with eigenvalues $\theta_1 = 2$, $\theta_2 = -1$. Condition (1.1) is easily seen to hold, since the only prime that divides the determinant of M is 2, and $(1,1)M^n = (1,1) \pmod 2$.

This subshift is obtained from the period doubling subshift by replacing each 2 with a pair of 1's. The tiling spaces built from this subshift are topologically conjugate to the tiling spaces of the period doubling subshift, if we take the same values of t_0 and let $t_1 = t_2/2$. In particular, the $\mathbb{R}$ action is weak mixing if t_0/t_1 is irrational, but is not topologically mixing. This implies that $\liminf(b(n) - a(n))$ is finite, so the $\mathbb{Z}$ action is not topologically mixing.

We can obtain precise bounds on $\liminf(b(n) - a(n))$ from the correspondence with the period doubling subshift. Every MPD word is obtained from a period-doubling word by replacing each 2 with 11 and then possibly truncating leading or trailing 11's to a single 1. From period-doubling words of length 2^n we obtain MPD words of length $\lfloor 2^{n+2}/3 \rfloor$ with $\lfloor 2^{n+1}/3 \rfloor + 1$ 0's, of length $\lfloor 2^{n+2}/3 \rfloor$ with $\lfloor 2^{n+1}/3 \rfloor$ 0's, or of length $\lfloor 2^{n+2}/3 \rfloor + 1$ with $\lfloor 2^{n+1}/3 \rfloor$ 0's. One can also obtain MPD words of length $\lfloor 2^{n+2}/3 \rfloor$ with $\lfloor 2^{n+1}/3 \rfloor - 1$ 0's from period-doubling words of length $2^n - 1$. Thus $b(\lfloor 2^{n+2}/3 \rfloor) = \lfloor 2^{n+1}/3 \rfloor + 1$ and $a(\lfloor 2^{n+2}/3 \rfloor) = \lfloor 2^{n+1}/3 \rfloor - 1$, so $\liminf b(n) - a(n)$ is at most 2.

Fix t_0 and t_1 with irrational ratio, and consider any tiling of $\mathbb{R}$ associated with the MPD substitution. Denote by Λ the set the endpoints of its tiles. It provides an interesting example of a uniformly discrete, relatively dense subset of the real line. The arithmetic difference $\Lambda - \Lambda$ is just the projection of $\Gamma - \Gamma$ along an irrational direction. Since the width of $\widetilde{\Phi}$ has infinite lim sup and finite lim inf, we obtain that $\Lambda - \Lambda$ is not uniformly discrete, but it is not eventually dense in $\mathbb{R}$ either. A subset

of $\mathbb{R}$ is said to be *Meyer* if it is relatively dense and its arithmetic self-difference is uniformly discrete. Thus, we obtain the following.

Corollary 5.3. *There exists a "binary" (i.e. with two possible distances between consecutive points) non-Meyer set $\Lambda \subset \mathbb{R}$ such that $\Lambda - \Lambda$ is not eventually dense.*

Open questions and directions for further research.

1. Is there a combinatorial description of substitutions on two symbols that have $|\theta_2| = 1$ and satisfy (1.3)? Are there primitive integer matrices with the second eigenvalue of magnitude one, which determine topological mixing (or lack thereof) for every substitution having this matrix?

2. What happens for substitutions on more than two symbols? Of course, there are trivial examples arising from the fact that a substitution may be recoded using a new, larger alphabet. With some caution, we can put forward the following *Conjecture: for a primitive substitution on any number of symbols, assuming none of the eigenvalues has magnitude equal to one, topological mixing is equivalent to weak mixing.* This concerns both the $\mathbb{Z}$ and $\mathbb{R}$ actions associated with the substitution.

3. What happens in higher dimensions? We do not know any examples of topologically mixing substitution $\mathbb{Z}^d$ or $\mathbb{R}^d$ actions for $d \geq 2$. In particular, is the pinwheel tiling system studied by Radin and coauthors (see [12]) topologically mixing? A well-known open problem is to determine whether the pinwheel system is strongly mixing. Checking topological mixing may be a more realistic goal.

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